

Electric-magnetic duality beyond four dimensions and in general Relativity *

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Abstract

After reviewing briefly the classical examples of duality in four dimensional field theory we present a generalisation to arbitrary dimensions and to p-form fields. Then we explain how U-duality may become part of a larger non abelian V-symmetry in superstring/supergravity theories. And finally we discuss two new results for 4d gravity theory with a cosmological constant: a new exact gravitational instanton equation and a surprising linearized classical duality around de Sitter space.

1 Electric-magnetic duality and self-duality

1.1 Gauge fields

The discrete (Z_4) and continuous ($SO(2, R)$) invariances of the Maxwell equation and of the gauge fixed Maxwell action [1] are a remarkable feature of 4 dimensional electromagnetism in vacuum. The inclusion of matter requires non trivial topology (like a possibly nontrivial $U(1)$ principal bundle) in order to preserve these symmetries. At the quantum level the lattice of electric-magnetic charges breaks the symmetry down to a discrete one. The Dirac-Schwinger quantization condition constrains the possible charges of a pair of dyons $D(e, g)$ and $D'(e', g')$ to satisfy:

$$4\pi(eg' - e'g)/h = \text{integer} \quad (1)$$

The two helicities of the electromagnetic field correspond to self-dual and antiself-dual field strengths. In euclidean signature the (real) classical

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field strength can be decomposed locally into the sum of a self-dual part and an anti self-dual part ie

$$dA \equiv F, F_{\pm} = \pm * F_{\pm} \quad (2)$$

where $*$ is the Hodge dualization operator on two forms. For Yang-Mills fields there is a celebrated generalization of the self-duality projection namely the instanton equation. Note that the usual instanton equation is first order and provides only special solutions to the full (vacuum) Yang-Mills equations.

1.2 Gauge forms

Pointlike electric charges are minimally coupled to “vector” potentials and the generalization for scalar fields resp higher $(p+1)$ -form potentials is their coupling respectively to instantons and p -branes. Abelian self-duality is possible in even dimensional spacetimes of dimension $(2p+4)$ of the appropriate signature for a single $(p+1)$ -form potential. There is a generalization of the quantization condition (1) to this situation as well and interestingly it involves a plus sign rather than a minus sign in $(4k+2)$ dimensions [2].

One key property of these remarkable self-dual solutions is that they minimize the action by saturating a topological charge bound: the so-called BPS bound. It is E. Bogomolny who analyzed systematically this mechanism and applied it to magnetic monopoles and dyons (independently studied by M. Prasad and C. Sommerfield) . The lower bound is typically a characteristic (for instance Pontryagin) number of the principal bundle under study [3].

2 U-duality: selecta

2.1 Gravity case

The Einstein action in D dimensions is invariant under diffeomorphisms of the manifold M_D . Upon dimensional reduction by r commuting one parameter isometry groups the effective action on the $(D-r)$ quotient space (of orbits) the set of equations becomes invariant under a group of internal symmetries that grows with r . Part of it is expected for instance $GL(r, R)$ or at least $SL(r, R)$ but other parts of it come as surprises, the first of which is the so-called Ehlers symmetry $SL(2, R)$ that is easy to verify after reducing ordinary Einstein gravity in $D = 4$ by one dimension ($r=1$). More generally reduction of pure gravity from D to 3 dimensions leads to a generalised Ehlers symmetry $SL(D-2, R)$, see for instance [4]. This is a major mystery and constitutes one of our motivations to concentrate on dualities in general, to discover new ones and to study their properties.

2.2 The supergravity magic triangles

If one considers at first the internal symmetries (commuting with the Poincaré group) one encounters often coset spaces, even Riemannian symmetric spaces, on which these symmetries act as real Lie groups. These

cosets are the target spaces where scalar fields (ie 0-forms) take their values. The symmetries are called U-dualities for historical reasons [5], approximately half of them act by (Hodge) dualities on the p-forms in their self-duality dimension. A remarkable collection of (pure in D=4) supergravity theories as well as their dimensional reductions down to 3 dimensions and their higher dimensional ancestors fit into a triangle with partial symmetry under the exchange of the space-time dimension with the number of supercharges see [4]. These groups are expected to play an important role in string theory after being broken down to a discrete (arithmetic) subgroup.

In the example of 4 dimensions for instance the U-duality group of maximal supergravity is the split real form of E_7 it contains a parity conserving subgroup $SL(8, R)$ and the other generators are dualities. The maximal compact subgroup of this real form of E_7 is $SU(8)$ sometimes called R-symmetry just to confuse us. The string “gauge group” is expected to be the intersection of the split E_7 with the discrete group $Sp(56, Z)$. E_7 is indeed a subgroup of $Sp(56, R)$. One must double the number of vector potentials from 28 to 56 to realize locally the action of dualities, it turns out that the doubled set of fields obeys first order equations that are now equivalent to the second order original equations. We shall recognize this phenomenon as rather general and this will lead us to V-dualities. The doubled set of first order equations is nothing but a (twisted) self-duality condition. For an early discussion of doubling see for instance [6].

$$E.F = S * E.F \quad (3)$$

In our example F is the 56-plet of field strengths, E is a representative of the scalar fields taking their values in the exceptional group E_7 and written in the **56** representation and S is a pseudo involution of square ± 1 that compensates for the square of the Hodge operation $** = \pm 1$.

3 V-duality

3.1 del Pezzo surfaces and Borcherds algebras

Another mystery of duality is the occurrence not only of the exceptional group E_7 but of the full (extended in fact) E series: $E_8, E_7, E_6, E_5 = D_5, E_4 = A_4, E_3 = A_2 \times A_1 \dots$ both as the U-duality groups of maximal supergravity reduced to 3,4,5,6,7,8... dimensions and as symmetry groups of type II string theories after torus compactifications. The equally mysterious occurrence of the E groups or rather of their Weyl groups acting on the middle cohomology of the so-called del Pezzo complex surfaces may be a related phenomenon. There are in fact two candidates for E_1 so let us choose A_1 which is known to be associated to the trivial bundle $CP^1 \times CP^1$ (one of the two “minimal del Pezzo surfaces”). $SL(2, Z) = A_1$ is known to be also the U-duality group of type IIB superstring theory in 10 dimensions (the top dimension). Besides the information provided by algebraic geometers (Y. Manin...) we used [7] one important remark of C. Vafa and collaborators who stressed the importance of rational cycles within the second cohomology of the del Pezzo complex surfaces. For instance

in the case of $CP^1 \times CP^1$ the middle cohomology is quite boringly equal to $Z + Z$, yet one axis of this lattice is selected by the complex geometry to be the root lattice of the above mentioned A_1 and the correspondence between spheres on the del Pezzo surface and D-branes on the string side [8] suggested to us that one should combine the Weyl cone of A_1 and the Mori cone of the cohomology into a Borchers cone associated to the simple (positive) roots of a generalized Cartan matrix obtained from that of A_2 by replacing one of the diagonal elements (2) by a zero! The correspondence is best understood in this case but more generally it is still useful [7]. The intersection form on the surface is in this case the metric on the Cartan subalgebra of a Borchers algebra.

3.2 Truncated Borchers algebras and V-duality

On the string/supergravity side we have known for a while [9] that there is a natural generalization of the Borel subgroup of U-duality (isomorphic to the corresponding non-compact symmetric space and target of the scalar fields) to a solvable group encompassing all the p-forms and encoding their non linear couplings but not the graviton field yet. The question was to give a name to this solvable group despite the absence of any reasonable classification of non semi-simple Lie algebras. It generalizes the encoding of nonlinear sigma model fields' couplings within the structure constants of a group, to that of higher forms' couplings in the (super)group structure of this solvable algebra. A (p+1)-form will have degree (p+1) and the Z-graded solvable superalgebra reduces in degree zero precisely the U-duality algebra or if one prefers its Borel subalgebra. There is a remarkable correspondence between the del Pezzo data and the string/M-theory data [7]. Two steps are left to ascend: firstly one should include gravity which only trickles down into this formalism after dimensional reduction, and secondly one must incorporate the fermions (this will require the enlargement of the Borel algebras to full V-duality symmetry groups in order to allow for their "maximal compact subgroups" whatever this means to act on the fermions, but we have lots of experience even in the infinite dimensional case of spacetime dimension 2).

4 Λ -Instantons

4.1 Gravitational instantons

Let us consider now a 4 dimensional Riemannian manifold and its Riemann curvature 4-tensor R . It is well known [3] that one may impose (Hodge) self-duality on the first (or second) pair of indices, this defines the usual gravitational instantons which are necessarily Ricci flat and provide a nice subset of solutions of the second order Einstein equations. One may also require to have double self-duality exactly as in (3)

$$R = S * R \tag{4}$$

where S is the dualization on the first pair of indices if * is the dualization on the second pair. This is equivalent to the Einstein space condition

(with unspecified cosmological constant). There is the conformal self-duality equation too that guarantees the existence of a twistor space see for instance [10].

4.2 Λ -instantons

It seems to have gone unnoticed that there is yet another equation for any given value Λ of the cosmological constant that provides what we call Λ -instantons [11]. It is obtained by adding in the ordinary gravitational instanton equation to the Riemann curvature tensor the combination

$$-\Lambda/3(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}) \quad (5)$$

the resulting tensor $Z_{\mu\nu\rho\sigma}$ turns out to be equal to the MacDowell Mansouri tensor associated to a de Sitter bundle [12]. The Λ -instanton equation reads simply

$$Z = *Z. \quad (6)$$

It implies the Einstein equation for that particular value of the cosmological constant but it is not equivalent to it.

5 Duality in the gravitational sector

5.1 Near flat space

In a nice paper [13] the dual form of 4d linearized Einstein gravity was found to be again of the same type. The authors introduced 2 prepotentials and their associated pregauge invariances beyond diffeomorphism symmetry and showed they were interchangeable by a continuous duality rotation on shell. Even off shell the non-covariant action is invariant under duality exactly as in the Maxwell case. Such a duality exists at the nonlinear level in the presence of one Killing vector field it is precisely the Ehlers symmetry, whereas such an isometry is not assumed anymore here. The prepotentials are defined by solving the hamiltonian and momentum constraints.

5.2 Near de Sitter space

It maybe encouraging to go beyond this linear truncation to linearize around a different background and to try and see whether such a duality symmetry persists. Around de Sitter space (but the sign of the cosmological constant is not really important for local questions) indeed the duality rotation exchanges the relevant components of the modified curvature tensor Z , the electric part is Z_{0m0n} and the magnetic part $1/2 Z_{0m}^{pq} \epsilon^{pqn}$. When the cosmological constant tends to zero the near flat space result is recovered smoothly.

6 Conclusion

We must now go nonlinear and it seems natural to expect from M-theory considerations that the dual theory does exist and that it is worth our

efforts. More specifically the dual diffeomorphism invariance is suggestive of a doubling of spacetime, allowing for some self-duality condition that reduces the effective dimension to 4. This doubling is very familiar in string theory. We had no time to review quantum effects like quantum anomaly or NUT charge quantization.

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